A STUDY OF THE STABILIZATION OF AN EARTH SATELLITE

by

Francois A. Jazede

Report No. PIBMRI-1079-62

for

Office of Ordnance Research
Box CM, Duke Station
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MRI

POLYTECHNIC INSTITUTE OF BROOKLYN
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ABSTRACT

This study is concerned with the stabilization in attitude of an earth satellite. In particular two ways of obtaining good stabilization are studied. One by use of a gas rocket; which yields a non linear system ("On -Off" type) studied by the phase plane method. The other by use of flywheels: the stabilization is then a linear one. In both cases it is shown that a great simplification can be introduced in the stability equations by the introduction of some approximations. Even, in the non linear case, the approximations (practically always justified) brin an enormous simplification in the computations by supressing the coupling between the three axis around which the stabilization is applied.

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CHAPTER I

Introduction

The goal of this thesis is the study of the attitude stabilization of an earth observation satellite. The orbit is assumed circular or near circular (eccentricity $e < 10^{-1}$); and the altitude is above 200 miles. in which case the atmospheric drag is negligable. Also, the perturbation torques are assumed very small (and indeed they are), compared to the inertial torques on the satellite - and in doing so it will be possible to linearize the equations of stability (small angles approximations: $\cos \theta \sim 1$, $\sin \theta \sim \theta$). The case of the insertion into orbit where large errors appear will not be studied.

After a first part, devoted to the study of the orbit from a mechanical point of view, the main part of this report will discuss the stabilization of the satellite by use of rockets (Chapter III).

The corrective torques needed for stabilization and produced by the rockets will be of the "on-off" type, with deadzone and time delay (in practice this is a good approximation for most cases).

The study of such a system in three dimensional space is rather complicated. But by the use of suitable approximations, it's possible to make the study in one dimensional space.

Then some particularities of the stabilization by a non-linear system will be studied "reticence behaviour" phenomena, limit cycle. It will appear that generally two orientations modes (coarse orientation mode, and fine orientation mode) are needed and can be obtained by switching the gain of an amplifier. Also, a quick study of the amount of stored gas for the rockets is made, for a satellite life of 8 to 10 months.

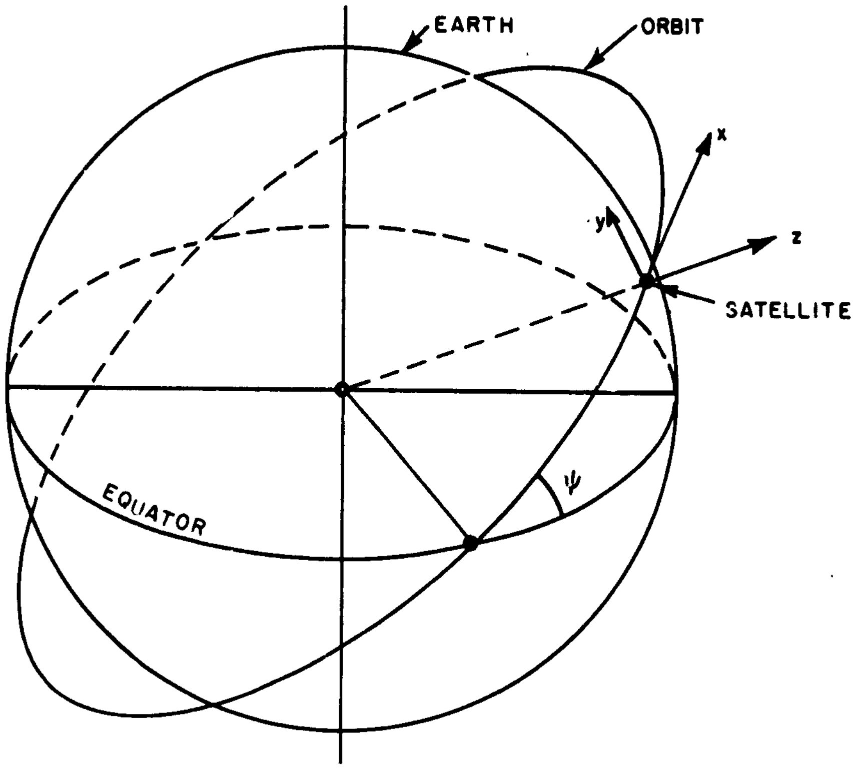
As such stabilization by rockets is not precise enough if the satellite carries cameras, a following chapter will be devoted to a flywheel stabilization system.

To conclude it can be said that both systems are needed: the rocket stabilization for large perturbations; and the fly wheel stabilization to achieve the stabilization within the required limits.

CHAPTER II

Mechanics of the Orbit

2.1 System of Reference*



System of Reference
Fig. 2. 1

a) X', Y', Z' are fixed-body axes in the satellite. For simplification sake they are chosen to coincide with the principal axces of inertia of the vehicle. The reduction in generality of the results, in so doing, is not serious, since most of the satellites have some sort of symmetry;and control will be desired about these principal axes.

The vehicle is built and oriented in such a way that (X', Y', Z') almost coincide with the reference system XYZ which we shall describe now.

b) If X, Y, Z (see Fig. 2.1) are chosen to be almost parallel to X', Y', Z' then the variation angles will be small, and we shall be able to linearize equations by the following approximations

$$\sin\theta \sim \theta$$
$$\cos\theta \sim 1$$

* See Reference (1)

So, as shown in Fig. 2.1, the Z axis is aligned to the geocentric vertical and taken positive outward from the center of the earth. The X axis is in the plane containing Z and the velocity vector, and oreinted in the direction of the velocity (for a perfectly circular orbit X and the velocity vector will coincide) Y is chosen to form a right-hand system with X, Z.

Such a reference system is not fixed in space, but rotates with an angular velocity Ω.

For a circular orbit we get

$$\Omega_x = \Omega_z = 0$$

$$\Omega_y = \Omega_o = \text{constant}$$

2.2 Disturbances Torques

Disturbance torques can be put into two categories, internal and external.

2.2.1 Internal disturbances are due to the motion of some internal piece, a shutter a rotating magnetic recorder,etc. and will not be taken into account for generally they are very small.

2.2.2 External disturbances can be divided into secular and cyclic types:

Secular disturbances

a) Aerodynamic drag which is of the order of 1 dyne cm at the altitude considered (at 200 miles). Above 200 miles aerodynamic forces are negligable, at least for a satellite life of 8 to 10 months.

b) Meteoritic collision for a meteorite of more than 10^{-3} gm, the probability of occurence is quite small*.

c) Earth's magnetic field

The torques are due to the eddy current effects or permanent magnet effects in the satellite which react with the earth field. These may be one of the most important sources of disturbances. They may be minimized by some special precautions. For-instance in the case of eddy currents, by using a skin of high resistivity or a skin made of small sections not electrically connected, also, internal current loops can be made by pairs of twisted wires whenever possible. The effects of the permanent magnets will be corrected by compensating coils along the 3 orthogonal axes. The order of magnitude of the magnetic torques is 100 dyne cm to 200 dyne cm.

d) Differential gravity

If a principal moment of enertial is not aligned with the yaw axis, a torque T_g is

*See Reference (2).

created. For a circular orbit and a small variation

$$Tg_x = \Delta I \, 3\omega^2 \, \theta_x$$

where

$$\Delta I = I_y - I_z$$

e) Earth's oblateness.

The oblateness of the earth is a perturbation which acts directly on the satellite orbit. The plane of the orbit oscillates slowly, the amplitude depending upon the inclination ψ of the orbit upon the equator. But such a perturbation is generally negligible.

Cyclic perturbations

a) Solar radiations

The solar radiation pressure is of the order of 5×10^{-5} dynes/cm^2.

b) Differential gravity.

Differential gravity effects the solar cells paddles.

c) Magnetic fields

For all of the above cyclic disturbances the frequency is about twice the orbital frequency (the period of the orbit being around 100 minutes for an altitude of 200 miles) and the maximum torque is some scores of dyne. cm (100 to 200 dyne. cm is a good average value). Finally only the action of the gravity gradient is simple enough to be accounted for simply in the equations of motion. The other perturbations being independant of the oreintation of the satellite, will be considered as those unavoidable disturbances which occurs in every serovomechanism. Their action will be that of an impulse (meteoritic impact), a step (magnetic disturbances), or a sinusoid (cyclic perturbations).

In any case they are shown to be small as far as their effects are concerned; and the perturbations on the stabilization will be well within the tolerance. For instance, a perturbation of 200 dyne x cm (which is high) will give the satellite an angular acceleration of

$$\ddot{\theta} = \frac{M}{I} = \frac{200}{10^8} \qquad (I > 10^8 \text{ g.cm.}^2)$$

$$\text{or } \ddot{\theta} \sim 10^{-6} \text{ rad./sec.}^2$$

which is negligible for the period of time we shall consider (we shall see that the oscillations have a period of about 100 sec.).

2.3. Equations of motion*

Two different sets of equations can be developed depending on how the correction

*See Reference (1) and (2)

torques are produced; by ejection of a certain mass of gas, or by an internal motion of a flywheel. If there are no moving parts in the satellite then the motion of the vehicle in orbit will be described by:

$$\bar{H} = \left[I \right] \bar{\omega} \qquad \qquad (2.1)$$
$$\bar{\omega} = \dot{\theta} + \Omega$$

Where Ω: angular velocity of the reference system
$\dot{\theta}$: angular velocity of the body w.r.t. the reference system
ω: angular velocity of the body w.r.t. the inertial space
w.r.t. : an abreviation for "with respect to".

and

$$\left[I \right] \equiv \begin{vmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{vmatrix}$$

Now if we call T any torque (disturbance or control acting on the vehicle) we have the relation

$$\frac{d\bar{H}}{dt} = T \qquad \qquad (2.2)$$

The expansion of equation (2.1) and (2.2) is:

$$\begin{aligned} I_x \ddot{\theta}_x + 4\Omega_o^2 (I_y - I_z)\, \theta_x + \Omega_o (I_x - I_y + I_z)\, \dot{\theta}_z &= T_{xc} + T_{xd} \\ I_y \ddot{\theta}_y + 3\Omega_o^2 (I_x - I_z)\, \theta_y &= T_{yc} + T_{yd} \qquad (2.3) \\ I_z \ddot{\theta}_z + \Omega_o^2 (I_y - I_x)\, \theta_z - \Omega_o (I_x - I_y + I_z)\, \dot{\theta}_x &= T_{zc} + T_{zd} \end{aligned}$$

A detailed derivation of these equations is given in Reference 1. These equations are valid for a circular (or near circular) orbit where $\Omega_x = \Omega_o = \text{constant}$
$\Omega_x = \Omega_o = 0$

T_{xc} is the control torque along x axis. T_{xd} is the disturbance torque along x axis. (note that the gravity-gradient torque is included in the left-hand side of the equations). If there are moving parts in the satellite (flywheels) then:

$$H_{syst.} = H_{veh.} + \sum (H_{wheels})$$

or, more explicitly:

$$H_{syst} = \left[K \right] \cdot \omega_{veh.} + J(\omega_{veh.} + \omega_{wheel}) \tag{2.4}$$

and $$\dot{H}_{syst.} = \bar{T}$$

where H_{syst} = Angular momentum of the whole system.

$[K]$ = Inertia of the satellite minus flywheels.

J = Inertia of a flywheel.

The expansion of (2.4) and (2.5) is given in references (see Ref. (3)). For the resulting equations see Chapter IV.

CHAPTER III

Non-Linear Stabilization Using Jets

We shall study the stability problem of a satellite using jets to create a control torque. The jets operate on-off, and the control system is nonlinear.

3.1 Nonlinear Stabilization System in the Three Dimensional Case.

a) Stability- In the absence of control torque the equations (2.3) after using Laplace transformed yield:

$$\left[I_x S^2 + 4\Omega_o^2 (I_y - I_z)\right]\theta_x + \Omega_o S(I_x - I_y + I_z)\theta_z = A(s) \qquad (3.1)$$

$$\left[I_y S^2 + 3\Omega_o^2 (I_x - I_z)\right]\theta_y = B(s) \qquad (3.2)$$

$$\left[I_z S^2 + \Omega_o^2 (I_y - I_x)\right]\theta_z - \Omega_o S(I_x - I_y + I_z)\theta_x = C(s) \qquad (3.3)$$

where A(s), B(s), C(s) stand for the Laplace transform of the perturbation torques and the initital conditions ($\theta_x(o)$, $\theta_y(o)$.....). The system will be stable if the roots of the following equations don't have a positive real part:

$$I_y S^2 + 3\Omega_o^2 (I_x - I_z) = 0$$

$$\begin{vmatrix} I_x S^2 + 4\Omega_o^2 (I_y - I_z) & S\Omega_o (I_x - I_y + I_z) \\ -S\Omega_o (I_x - I_y + I_z) & I_2 S^2 + \Omega_o^2 (I_y - I_x) \end{vmatrix} = 0$$

or

$$I_y S^2 + 3\Omega_o^2 (I_x - I_z) = 0$$

$$I_x I_z S^4 + S^2 (I_x \Omega_o^2 (I_y - I_x) + 4 I_2 \Omega_o^2 (I_y - I_z)$$
$$+ S^2 \Omega_o^2 (I_x - I_y + I_z)^2 + 4\Omega_o^4 (I_y - I_z)(I_y - I_x)$$

An evident solution is that

$$I_x - I_z > 0 \quad I_y - I_x > 0 \quad I_y - I_x > 0$$

or

$$I_y > I_x > I_z \qquad (3.4)$$

Such a relation is sufficient but not necessary.

b) Approximations; and reduction to the one dimension case

Assuming now that the inequality (3.4) is satisfied assuring that the system is stables the details of stability can be investigated. For example, from equation (3.3) we notice that the damping or rate term is non-existant; and a critical study of equations (3.1) to (3.3) will show that no damping effects are present. For example, the magnitude of the terms in Eq. 3.1 will be examined. The value of I_x, for a compact vehicle of several hundred pounds, is of the order of 10^8, 10^9 g cm^2 (for a compact satellite, which is generally the case, the variation of I from one principal axis to the other is no more than 1 order of magnitude -So, $I_y \sim 10^8$, $I_z \sim 10^8$, $\ddot{\theta}_x$ is determined from: $L = d.T = I_x \ddot{\theta}_x$ where T is the thrust of the control jets and d the distance between the jets. Usually T is of the order of some $\sim 10^{-3}$ dynes. With d=100 cm one obtains $\frac{L}{I} \sim 10^{-4}$, and $\ddot{\theta} = 10^{-4}$ rd/S^2. Also $\Omega_o = \frac{2\pi}{100 \times 60} \simeq 10^{-3}$ assuming a rotation of the satellite around the earth in 100 minutes. The quantity $(I_y - I_z)$ is of the order of I_x, at the most $\simeq 2$ or 3×10^{-1} rd and $4\Omega_o^2(I_y - I_z)\theta_x \simeq 10^{-6} I_x$ at the most. Also $\dot{\theta} < 10^{-2}$ rd/s, and in average $\theta \sim 10^{-3}$ rd/s (such values can be obtained from any articles in the litterature giving data of the performances of any satellite). Lastly, $\Omega_o(I_x - I_y + I_z)\theta_x < 10^{-5} I_x$.

It appears that all terms of equation (3.1) are negligible compared to $I_x \ddot{\theta}_x$. The same for (3.2) and (3.3). These equations can now be approximately written as:

$$
\begin{aligned}
I_x \ddot{\theta}_x &= A(s) + T_x(s) \\
I_y \ddot{\theta}_y &= B(s) + T_y(s) \\
I_z \ddot{\theta} &= C(s) + T_z(s)
\end{aligned}
\tag{3.5}
$$

T(s) is the Laplace transform of the control torques. The coupling can be neglected and each equation can be studied separately. Now, there is no rule which will allow one to choose T(s), but as it will be seen later, for the linear system, the form of T(s) needed will be: $T_x(s) \equiv A\theta_x + B\dot{\theta}_x$. Thus, a good first try for the nonlinear system is: $\alpha T_x(s) \equiv$ sign $(A\theta_x + B\dot{\theta}_x)$ or better $T_x(s) =$ sign $(A\theta_x + B\dot{\theta}_x + C)$, C being a deadzone which always appears in such systems. Nonlinear control and compensation is discussed in detail in the next section.

3.2 Study of a Nonlinear System in the One Dimension Case (by the phase plane method)

The block diagram of such a nonlinear gas jet system can be represented in the following way (See Fig. 3.1).

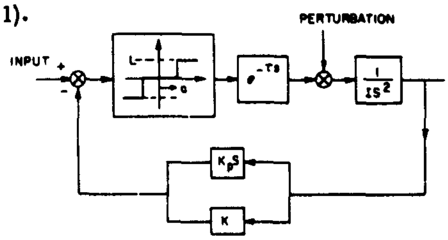


Fig. 3.1: Block diagram of a jet stabilization system (stabilization around one axis only)

The parameter 2a is the deadzone width which exists in most on-off systems. T is the sum of different delays which may occur in the loop between the detection of a deviation and the applied control correction. K_p is the gain of the rate gyro. K is the gain of the sun and earth sensor which detects any deviation θ. We call $\frac{K_p}{K} = \lambda$. Now as we have a stabilization system, the input will be zero (a correction of attitude commanded from the earth can be considered as a perturbation), and the loop can be considered in the following way (Fig. 32).

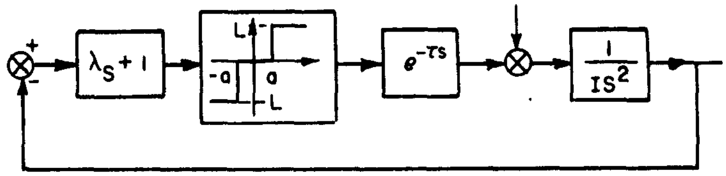


Fig. 3.2 Loop diagram for jet stabilization around one axis

To begin with, the perturbations will not be considered (they will be introduced later).

a) $\underline{\tau = 0}$

The open loop can be translated into the equation:

$$I \frac{d^2\theta}{dt^2} = L \text{ sign } (-\theta - \lambda \frac{d\theta}{dt} \pm a) \tag{3.6}$$

The phase-plane method will be used to study the system. Two new variables are introduced in (3.6)

$$x = -\theta$$

$$y = +\frac{dx}{dt} = -\frac{d\theta}{dt}$$

Then (3.6) becomes:

$$I \frac{dy}{dt} = L \text{ sign } (x + \lambda y \pm a) \tag{3.7}$$

Such a differential equation can be represented by a curve (C) in a plane of coordinates (x, y) called the phase plane. The representative point (x, y) of Eq. 37 will move on C.

When the sign of $x+\lambda y \pm a$ will change there will be a discontinuity of shape on C (whose differential equation will leap, for instance, from $\frac{I}{L} \frac{dy}{dt} = 1$ to $\frac{I}{L} \frac{dy}{dt} = -1$). Such discontinuities occur on the lines $x+\lambda y \pm a = 0$, which are called the switching lines.

The equation of curve (C) in the phase plane will now be examined. **Assuming,** at the start, that sign $(x+\lambda y-a)<0$, Eq. (3.8) becomes $I\frac{dy}{dt} = -L$.

or since $y = \frac{dx}{dt}, \frac{dy}{dt} = \frac{dy}{dx}, \frac{dx}{dt} = \frac{dy}{dx}\, y$

so
$$I\, y\, \frac{dy}{dx} = -L \qquad (3.8)$$

which yields by integration

$$Iy^2 = -2Lx + \text{Constant} \qquad \text{(parabola).}$$

Assuming that at $t=0$, $x=x_o$, $y=0$ then: $Iy^2 = -2L(x-x_o)$ (3.9)

and the time to coast along such a parabola is $t = -\frac{I}{L}\, y$ which is obtained by replacing $dx = y\,dt$ in (3.8). Assuming equation (3.9)

$$\frac{L}{I}\, t^2 = -2(x-x_o)$$

b) $\tau \neq 0$

Now, due to the delay τ, there is no switch on the line $y = \frac{-(x-a)}{\lambda}$. But the point (x, y) will move on the parabola (3.9) during τ more seconds, before the application of the torque takes effect (see Fig. 3.4).

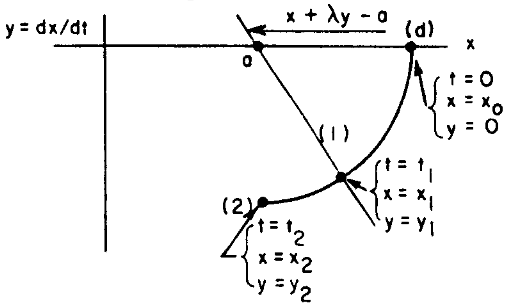


Fig. 3.4 Influence of a time delay τ on the switching line

(Due to the delay τ the switching occurs at (2) instead of (1)).

From the following equations:

$$\begin{cases} t_1 = -\frac{I}{L}\, y_1 \\ t_1^2 = -\frac{2I}{L}\,(x_1 - x_o) \\ x_1 + \lambda y_1 - a = 0 \end{cases}$$

We find

$$t_1 = -\lambda + \left(\lambda^2 + \frac{2I}{L}(x_o - a)\right)^{\frac{1}{2}} = -\lambda + \Delta$$

(the other value of t_1 is < 0 and so cannot be considered).
Now we know that $t_2 = t_1 + \tau$ or $t_2 = -\lambda + \Delta + \tau$.

Substituting the above value of t_2 in the following equations

$$\begin{cases} t_2 = -\dfrac{I}{L} y_2 \\ t_2 = -\dfrac{2I}{L}(x_2 - x_o) \end{cases}$$

yields the relation between y_2, and x_2 given in Eq. 3.10.

$$y_2 = \frac{x_2 - a - \dfrac{L\tau}{2I}(\tau - 2\lambda)}{(\tau - \lambda)} \qquad (3.10)$$

which is the equation of a delayed switching line. Eq. (3.10) is valid for a point (x, y) being in the fourth quadrant (or the first). Symmetrically, i.e., in changing a into $-a$ $\frac{L}{T}$ into $-\frac{L}{T}$ (Fig. 3.5 results)

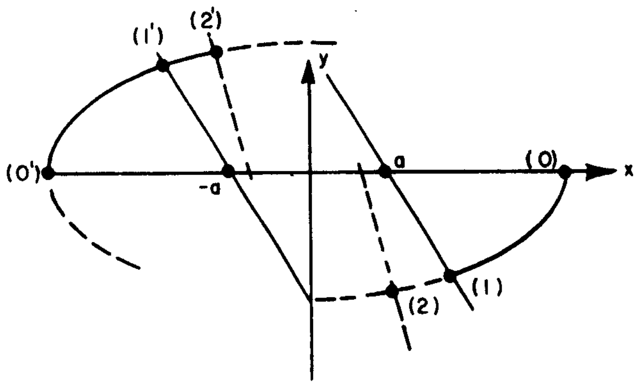


Fig. 3.5 Symmetry between the switching lines

We find the switching line for the second quadrant

$$y = \frac{x + a + \dfrac{L\tau}{2I}(\tau - 2\lambda)}{(\tau - \lambda)} \qquad (3.11)$$

After switching on the lines (3.10) or (3.11) the jets will be off, i.e. no control torque will be exerted and y (or $\frac{d\theta}{dt}$) will be constant. In the resulting motion the describing point (x, y) will move on a line parallel to the x axis. (see Fig. 3.6).

Due to the delay τ, the switch jets on command will occur τ seconds after point (3) in the phase plane is reached.

We have $$x_4 = x_3 + \tau\, y_4$$

and $$y_4 = y_3 = -\frac{x_3 + a}{\lambda}$$

from which we deduce $y_4 = \frac{x_4 - a}{\tau - \lambda}$ which is the equation of the third switching line. Changing a into $-a$ in this expression gives the 4th switching line.

Finally we have 4 delayed switching lines

$$\left\{\begin{array}{ll} y = \dfrac{x \pm a \pm \frac{L\tau}{2I}(\tau - 2\lambda)}{(\tau - \lambda)} & (3.12) \\ y = \dfrac{x \pm a}{\tau - x} & (3.13) \end{array}\right.$$

See Fig. (3-6).

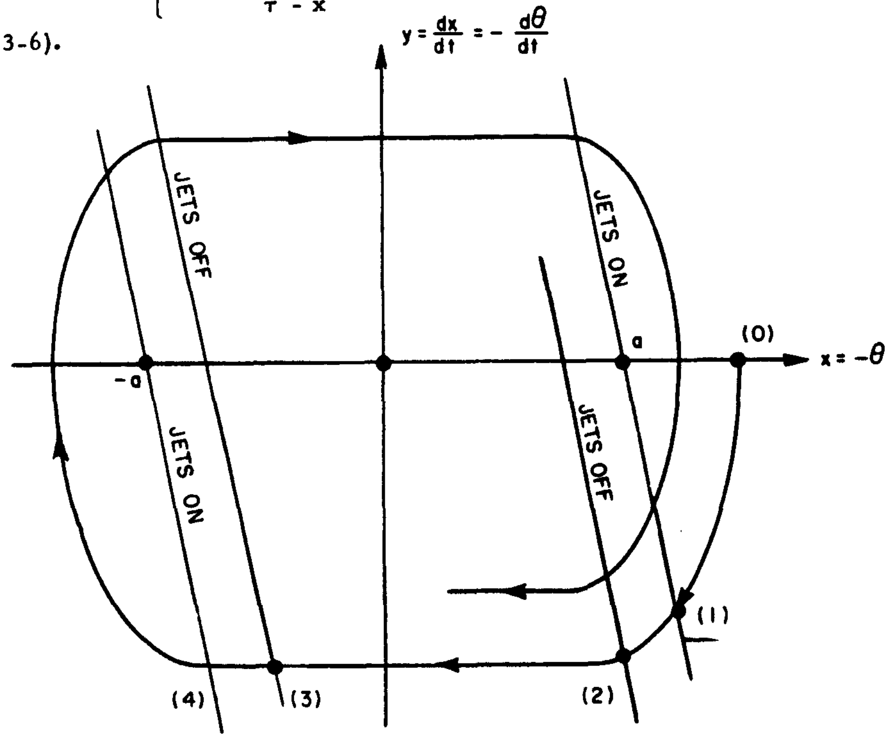


Fig. 3.6 Phase plane for the system $I\frac{dy}{dt} = L\,\text{sign}\,(x + \lambda y \pm a)$

The Fig. (3.6) makes clear that the slope of the switching lines must be < 0, if we don't want the describing point to diverge, i.e., an unstable system to occur. Therefore, stability requires $\lambda > 0$ since τ is always $<< \lambda$.

3.3 Particularites of a Jet Stabilization System - (The single dimentional case)

A study of Fig. 3.6 shows two particularities of such a system.

a) "Reticence" behavior (the use of such a term will become clear later).

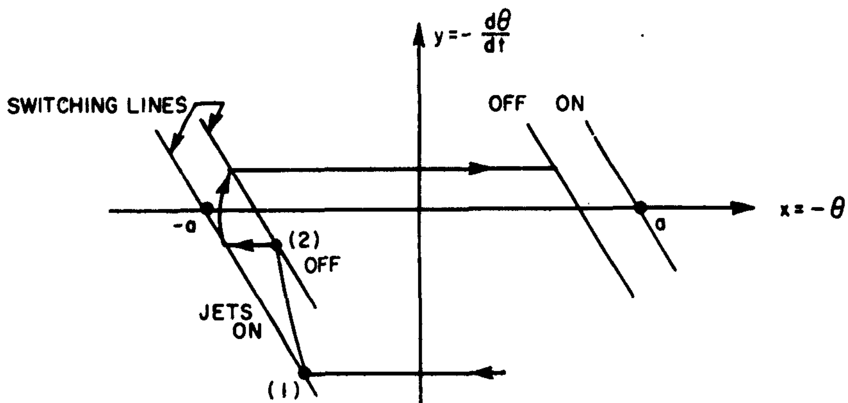


Fig. 3.7 Reticence behavior

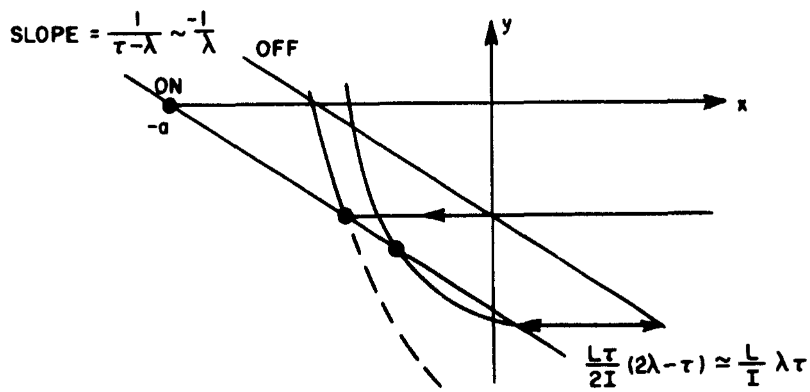


Fig. 3.8 Reticence behavior - its causes

As the trajectory coasts through the deadzone (see Fig. 3.7) and reaches a switching line, a negative torque is commanded (position (1) Fig. 3.7); $\frac{d\theta}{dt} = -y$ decreases but θ is still increasing due to the inertia of the satellite - and if λ is small the sign of $\theta + \lambda\dot{\theta} \pm a$ may change before $\dot{\theta}$ changes in sign. The rockets will then be switched off with a $\frac{d\theta}{dt}$ unchanged in sign (position (2)). So the same pair of jets will again be switched on (position (3)) after a certain increase of θ etc. In short, the same pair of jets changes the direction of rotation of the satellite after many on cycles and not after one unique big thrust.

This way of stabilizing the satellite may seem economical, for instead of oscillating around the equilibrium position the satellite comes very near to it, very quickly. It is important to study, however, what the conditions must be imposed on the control system, in such a case.

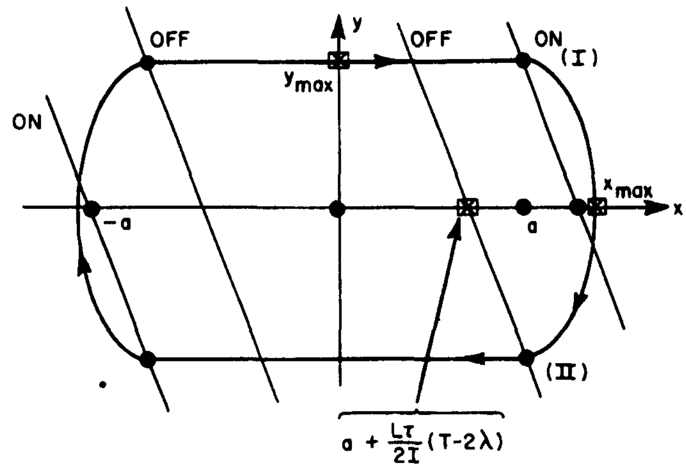


Fig. 3.9 Limit cycle behavior

The graph (3.8) shows that at some point the |slope| of the parabola must become greater than the |slope| $\frac{1}{\lambda}$ of the switching line. This point must be in the range of values of $y < 2 \times 10^{-2}$ rd/s (a greater value can come only from a bad insertion - accident which is not taken in account) and $y < 10^{-3}$ rd/s (below such rotation no control is needed: for such small rotation is not a trouble for the observation instruments of the satellite.)

Finally: |slope| of parabole $= \frac{L}{I}\frac{1}{y}$

and we want

$$\frac{1}{\lambda} < \frac{L}{I}\frac{1}{y}$$

or

$$\frac{L}{I} > \frac{|y|}{\lambda}$$

The lowest value of y is around 10^{-3}

$$\lambda \sim 10$$

so

$$\frac{L}{I} > 10^{-4}$$

But in most cases, $\frac{L}{I} \sim 1$ or 2×10^{-4} rd|sec .

So the reticence behavior will occur only for very small values of y when, as we shall see later, the jet must give way to flywheels.

b) Limit cycle behavior

Fig. 3.7 shows that the point (x, y) is becoming "nearer and nearer" to the origin but due to the symmetry of the switching lines it may fall into what is called a "limit cycle", i.e., the point (x, y) will endlessly follow the same trajectory around the origin. See Fig. (3.9).

The coordinates of points (I) and (II) are:

$$X_1 = X_{II} = \frac{1}{2}\left[a + a + \frac{L\tau}{2I}(\tau - 2\lambda)\right] \tag{3.14}$$

$$Y_1 = -Y_{II} = \frac{X_1 - a}{\tau - \lambda} \equiv y_{max}$$

or

$$Y_{max} = \pm \frac{L\tau}{4I}\frac{(\tau - 2\lambda)}{(\tau - \lambda)} \tag{3.15}$$

X_{max} is such that

$$y^2 = -\frac{2L}{I}(X - X_{max})$$

in which x and y are replaced by x_I and y_{II} to find X_{max}.

The result is

$$Y_{max} = \pm \frac{L}{4I}\beta$$

$$\pm X_{max} = a + \frac{L}{4I}\left[\frac{\beta^2}{8} + \beta(\tau - \lambda)\right] \tag{3.16}$$

$$\text{with } \beta = \frac{\tau(\tau - 2\lambda)}{\tau - \lambda}$$

Note that $\tau \sim 10^{-2}$ sec
$\lambda \sim$ 5 to 15 sec $\gg \tau$

so $\beta = 2\tau$

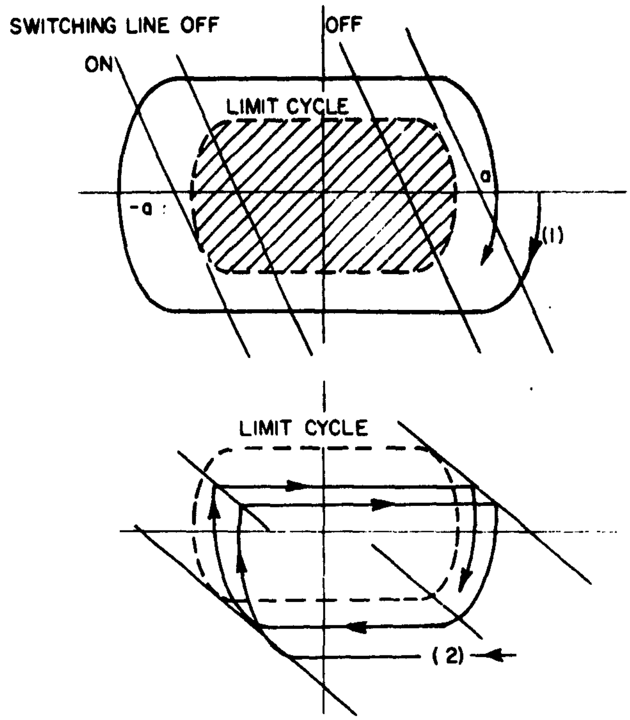


Fig. 3.10 Two ways of approaching the limit cycle (1) staying "outside" of it and (2) Moving inside and outside the limit cycle.

A graphical study (see Fig. 3.10) shows that the trajectary approaches the limit cycle in the steady state. It may sometimes "enter" the zone delineated by the limit cycle curve, but it will get out quickly and start again approaching such limit cycle.

3.4 Study of a Perturbation on the Rocket Stabilization System

This paragraph will be devoted to the study of the damping of x and y versus time, i.e., the finding of a relation between x (and y) at the points (0), (6) on Fig. 3.11, and the time. (The variation of x (or y) between points (0) and (6) can be considered as the variation driving a psuedo period).

Indeed the perturbation started at a point (x_i, y_i) somewhere in the plane; but it's more practical to start from a particular point for instance from a point on a switching line); and to go back to (x_i, y_i) from the chosen particular point by a change in the time origin. The following computations will give the values of the coordinates of points (o), (1), (2); from there it will be possible to deduce easily 3, 4, 5 by use of the sym-

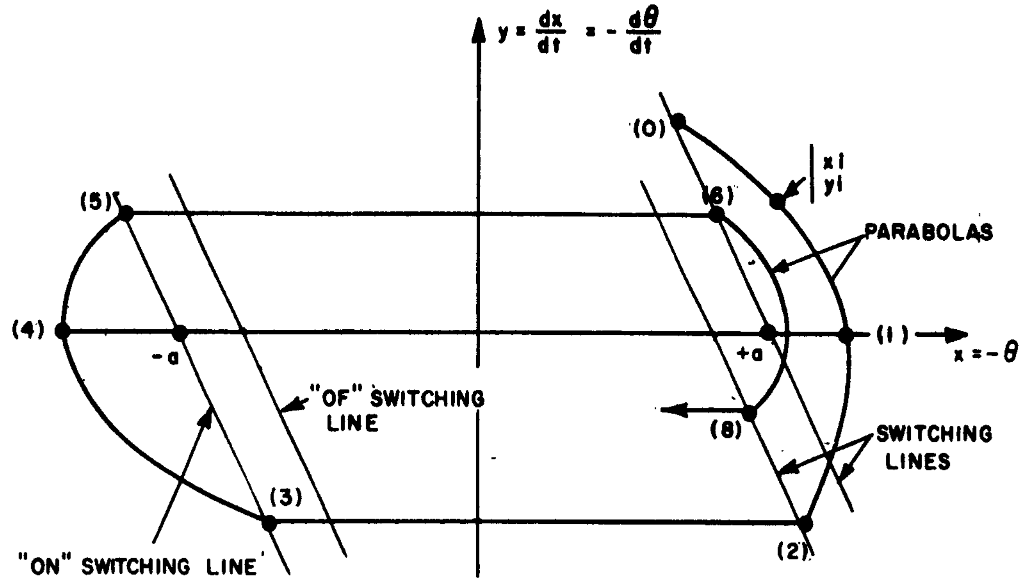


Fig. 3.11 Phase Plane for $x(=-\theta)$ and $y(=-\frac{d\theta}{dt})$

metry in the Fig. 3.11. The following equations will be used:

parabola $$y^2 = \pm \frac{2L}{I}(X - X_o) \qquad (3.19)$$

switching line $$y = \pm \frac{-x-a}{\lambda - \tau} - \frac{L\tau}{2I}\left(\frac{2\lambda - \tau}{\lambda - \tau}\right) \qquad (3.20)$$

which can be approximated by $y = -\frac{x-a}{\lambda}$ since τ is generally one hundred times less than λ, and since the 4 switching lines can be approximated by 2 switching lines only. The time along a parabolic path varies as given by the equation

$$t = +\frac{I}{L}|y| \qquad (3.21)$$

The intersection of a switching line (which replaces two switching lines almost in coincidence) and of a parbola yields the points (0) and (2), or from Eq. (3.19) and (3.20) it turns out:

$$y_o = \frac{L\lambda}{I} - A$$
$$y_2 = \frac{L\lambda}{I} + A$$

(λ is known to be negative)

with $$A = \left[\left(\frac{L\lambda}{I} \right)^2 - \frac{2L}{I} (a - x_1) \right]^{\frac{1}{2}}$$

[Note that A is smaller than $|\frac{L\lambda}{I}|$ for x_1 greater than a]

Also (3.22) yields

$$y_{.1} = \frac{L\lambda}{I} + A$$

Form Eq. (3.21) the time to move from point (0) to point (2) is $t_2 - t_o$ such that $t_2 - t_o = +\frac{I}{L} (|y_o| + |y_2|) = +\frac{I}{L} (y_o - y_2)$.

The final result is:

$$y_o + y_2 = \frac{2L\lambda}{I} \tag{3.22}$$

$$t_2 - t_o = +\frac{I}{L} (|y_o| + |y_2|) \tag{3.23}$$

$$= \frac{2IA}{L} \tag{3.23bis}$$

From there the graph of y can be drawn. For if y_o is known (and it is known from the initial perturbation y_i) Eq. (3.23) gives y_2 and then (3.23) gives the time elapsed between the values y_o and y_2; and (3.21) shows that y is linear in t. From there point (3) is deduced:

$$y_3 = y_2$$

$$t_3 - t_2 = \frac{2a}{y_2}$$

since $$\frac{dx}{dt} = \text{constant} = y_2$$

a integration yields: $$t = \frac{\text{variation of } x}{y_2} = \frac{2a}{y_2})$$

But the quantity $t_3 - t_2$ is negligable compared to $t_2 - t_o$ (a is of the same order of magnitude as y_2); so the 2 points (2) and (3) will be in coincidence when drawing the graph ot y versus time. As for the points (3), (4), (5) they are deduced from the preceding computations merely by replacing a by -a, $\frac{L}{I}$ by $-\frac{L}{I}$ and the point (0), (1), (2) respectively by (5), (4), (3) i.e., it turns out that:

$$y_5 + y_3 = -\frac{2L\lambda}{I} \tag{3.24}$$

$$t_5 - t_3 = \frac{I}{L} (|y_5| + |y_3|)$$

and $y = \frac{L}{I}$ t, from Eq. (3.21), i.e., y is varying linearily with respect to time, and the

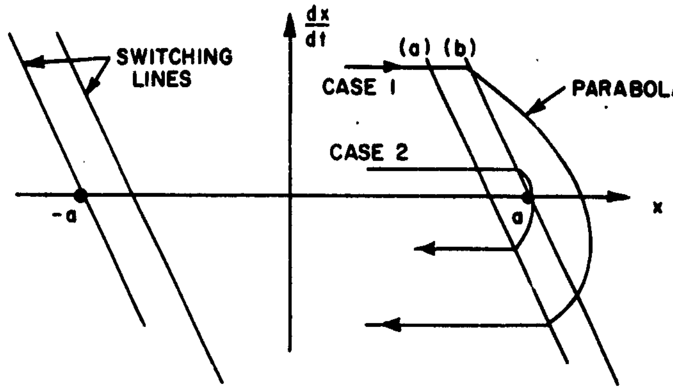


Fig. 3.12

slope of y versus t is $\frac{L}{I}$.

From point (5) to (6) y remained constant, but the time $t_6 - t_5$ is negligable (being equal to $\frac{2a}{y_5}$).

Finally it appears that the graph of y versus time will be made of segments of lines, of slopes $\pm \frac{L}{I}$, and the extremities of such segments will be given by Eq. (3.22) or (3.24). See Fig. 3.13. Note that the units in Fig. 3.13 has been normalized, i.e., instead of y and t, $y = y.\frac{I}{2L\lambda}$ and $\frac{t}{2\lambda}$ are used; the slopes of the lines becom then ± 1, and the difference between 2 maxima of y is ± 1; and the time elapsed between 2 maxima of y is equal to the sum of the absolute values of these maxima. It must be noted that when a positive maximum of Y is less than 1, then the following maximum of Y should be positive in order to satisfy Eq. 3.22; that is in contradicition with what is shown in Fig. 3.11 where 2 consecutive maximum of y (and so of Y) are of opposite signs. This contradiction is due to the fact that the approximation, made in assuming 2 switching lines in coincidence is no longer valid as it appears in the following Fig. 3.12.
Case one the 2 switching lines (a) and (b) can be assumed in coincidence. Case two such an approximation is no longer valid.

The integration of $y = \frac{dx}{dt}$ yields x. In fact from: $Y = \frac{dx}{dt} = \frac{dx}{2\lambda dT} = Y . \frac{2L\lambda}{I}$
the integration gives

$$x = \frac{4L\lambda^2}{I} \int_0^T Y \, dT$$

and since Y is linear in T, $\int_0^T Y \, dT$ is the equation of a parabola. Being careful about the concavity of the parabola, it's easy to draw the graph of x versus T. (See Fig. 3.14).

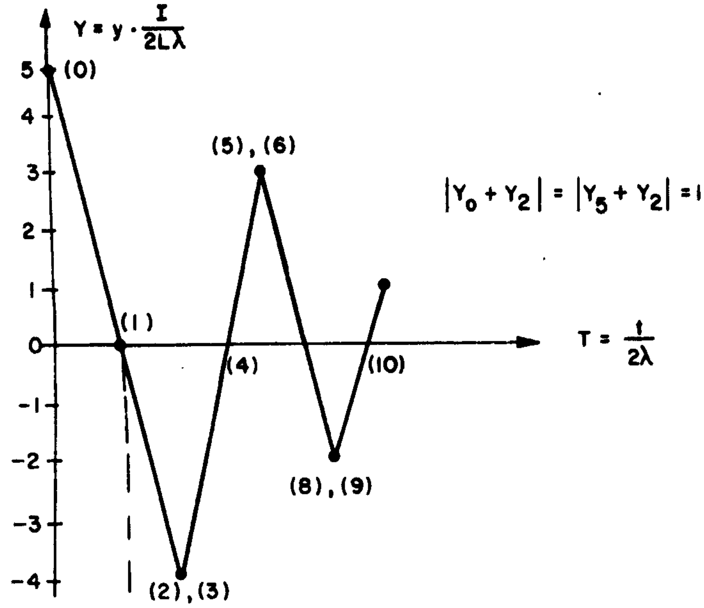
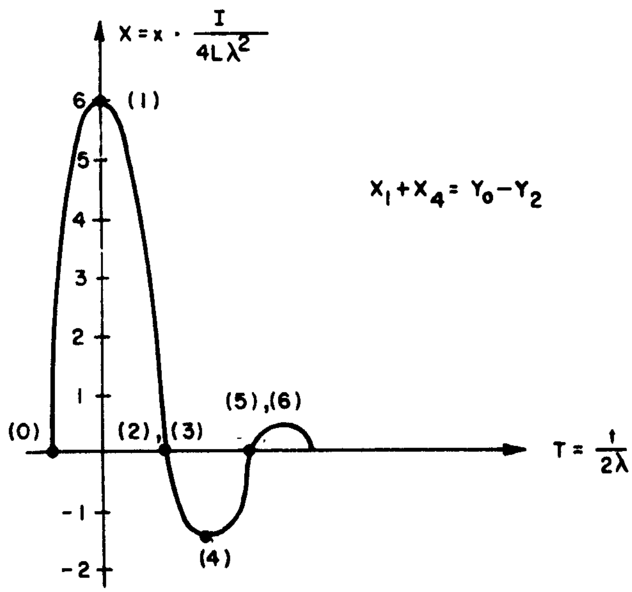


Fig. 3.13 Damping of the Satellite Angular Velocity Versus Time



For an explantion of the points (1), (2) etc. see Fig. 3.11.

Fig. 3.14 Damping of the Satellite Angular Error Versus Time

The two figures of x, and y versus t have been drawn one under the other, because it is easy to sketch one graph, knowing the other. Note that, exactly, the portions of parabola of Fig 3.14 should be recorded by straight lines originated by the integration of y=constant (See Fig. 3.13); but is has been shown that such constant values of Y are maintained during too short a time to be taken into account.

It must be noted also that the parabolas are symetrical around the Y axis; and that they are connected between themselves on the T axis (this is shown by Fig. 2.11, where the change of sign occurs on the horizontal lines, between 2 parabolas. The maxima of X are computed from the graph of Y versus T.

The 2 curves 3.13 and 3.14 give some conditions on the stability parameters. For, to have a quick damping of a perturbation:

λ should be large (so for a given T, t will be small)

$\frac{I}{2L\lambda}$ should be as small as possible.

The preceding study was made to get an idea of the damping of θ and $\dot{\theta}$ when the perturbation is an impulse. The system is stable, and provided a good choice of the parameter $\frac{I}{L}$, λ etc... the damping will be fast enough. For other types of perturbations (step, sinusoid) the phase plane method is not very adequate. Nevertheless, if the perturbations are not too big (i.e., some 100 dync. cm) they can be compared to an impulse: for under such disturbances (a step θ and $\dot{\theta}$ will be built up very slowly; and beyond a certain deadzone the stabilization system will act swiftly enough so that it's possible to neglect the small influence of the step disturbance on θ and $\dot{\theta}$.

3.5 Energetical Study

When the angular velocity is varying from 0 to y, the energy used is

$$E = \frac{1}{2} I y^2$$

Forinstance, in the case of the Fig. 3.13 it will be:

$$E = \frac{1}{2} I y_o^{\ 2} + I y_2^{\ 2} + I y_5^{\ 2} + I y_7^{\ 2}$$

or since $Y = y \cdot \frac{I}{2L\lambda}$

$$\frac{I}{(2L\lambda)^2} \ E = \left[(\frac{1}{2} Y_o^{\ 2} + (Y_o - 1)^2 + (Y_o - 2)^2 + Y_o - 3)^2 \ \right]$$

$$= (\frac{7}{2} Y_o^{\ 2} - 11 Y_o + 17)$$

$$E = I \left[\frac{7}{2} y_o^{\ 2} - \frac{11 y_o (2L\lambda)}{I} + 17 \left(\frac{2L\lambda}{I}\right)^2 \right]$$

The energy used for a given perturbation will be smaller if: $\frac{I}{L}$ is large and λ small.

CHAPTER IV

Flywheel Stablilization

4.1 Equations of Motion

Referring to Chapter II and Reference 3 the following equations are derived for the motion of the vehicle and its flywheels

$$\left[\begin{array}{c|c|c} I_x S^2 + (I_y - I_z)\omega^2 + J\Omega_y\omega & J\Omega_z S & (I_x - I_y + I_z)\,\omega\, S - J\Omega_y S \\ \hline -J(\Omega_z S + \Omega_x \omega) & I_y S^2 & J(\Omega_x S - \Omega_z \omega) \\ \hline -I_x - I_y + I_z)\,\omega\, S + J\Omega_y S & -J\Omega_x S & I_z S^2 + (I_y - I_x)\,\omega^2 + J\Omega_y \omega \end{array}\right] \left[\theta\right]$$

$$+ J \begin{bmatrix} S & 0 & \omega \\ 0 & S & 0 \\ -\omega & 0 & S \end{bmatrix} \left[\Omega\right] = \left[\bar{T}\right] \qquad (4.1)$$

With: I_x, I_y, I_z inertia of the satellite minus flywheels on the X, Y and Z axis.

J inertia of each flywheel.

Ω angular velocity of the satellite in the X, Y, Z frame.

ω angular velocity of the X, Y, Z frame with respect to an inertial frame.

T disturbaces torques.

Since the gravitational torques are known, T can be partitioned into two parts:

$$T = T_{dist.} + T_{grav.}$$

$$\text{with } T_{grav.} = \begin{vmatrix} (I_y - I_2)\omega^2 & 0 & 0 \\ 0 & (I_x - I_z)\omega^2 & 0 \\ \theta & 0 & 0 \end{vmatrix} \left[\theta\right]$$

In a very rough approximation of equations (4-1) it is possible to neglect almost all of the left-hand side terms of these equations. For instance, the first equation of (4-1) can be written as:

$$I_x \ddot{\theta}_x + \theta_x \left[(I_y - I_z)\, 4\omega^2 + J\Omega y\,\omega \right] + \dot{\theta}_y\, J\Omega_z \qquad (4.2)$$

$$+ \dot{\theta}_2 \left[(I_x - I_y + I_2)\omega - J\Omega_y \right] + J(\dot{\Omega}_x + \omega\,\Omega_z)$$

$$= T_{dist.}$$

"Reasonably" it can be assumed:

- $J\Omega << I$* (as $J \approx 10^{-5}$ I it would imply $\Omega << 10^{5}$ rd|S which is justified)
- $\dot{\theta} < 10^{-3}$ rd|S all greater values being reduced by the rocket stabilization.
- $\ddot{\theta} > 10^{-5}$ rd|S^2 which is true only in certain cases.
- $\omega \approx 10^{-3}$ rd|S

Then Eq. (4.2) can be rewritten as:

$$I_x \ddot{\theta}_x + J\dot{\Omega}_x = T_{dist.}$$

and the same for the 2 other coordinates, $J\dot{\Omega}$ is the control torque; it can be chosen as:

$$J\dot{\Omega} = K\theta + K_p\dot{\theta}$$

such choice being justified by the fact that it will appear convenient.

4.2 Study of a Flywheel Stabilization around One Axis.

With such approximations the coupling between axes disappears. From now on the stabilization around each axis can be studied separately. The block diagram of a stabilization by flywheel will be the same for each axis. It has been drawn in Fig. 4.1:

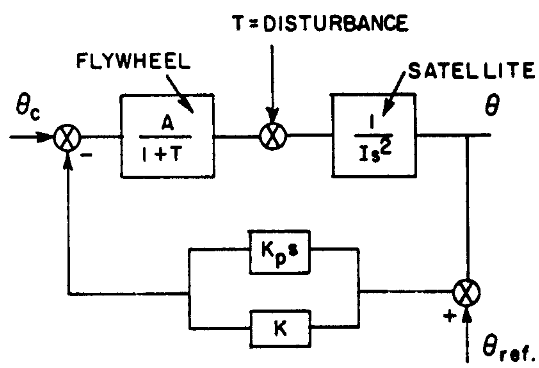


Fig. 4.1 Block diagram of a stabilization by flywheel system

*<< meaning " far less than".

θ_c: correction on the attitude satellite commanded from earth (in general $\theta_c = 0$).

θ_{ref}: is the correction on the orientation of the Z axis (see Fig. 2.1). It's due to the rotation of the satellite frame around the earth; and in case of a circular orbit $\theta_{ref.} = \omega t$, t being the time, and "ω" a constant. The transfer function of the flywheel has been taken: $\frac{A}{1+T_s}$. The flywheel being mainly an electric motor, this is the transfer function of some electric motor. But the justification is given in Appendix 4, and I is the inertia of the satellite (minus flywheel) around the axis (I stands for I_x or I_y or I_z). The transfer function of diagram (4-I) is easily found to be

$$\theta = \frac{1+TS}{IS^2(1+TS) + A(K_pS+K)} T_d + \frac{A(K_pS + K)}{IS^2(1+TS) + A(K_pS+K)} \theta_{ref.}$$

a) $T_d = 0$ (no disturbance), as the orbit is almost a circle:

$$\frac{d\theta_{ref}}{dt} = \omega = \text{Constant}$$

we want $\theta - \theta_{ref.} = 0$ for a ramp input of $\theta_{ref.}$

$$\varepsilon = \theta\,\text{ref.} - \theta = \frac{\omega}{K_v}$$

$$K_v = \lim_{s \to 0} S \cdot \frac{A}{JS^2(1+TS)} \cdot (K + K_pS) \to \infty$$

i.e., $\varepsilon = 0$

So no error is due to the steady variation of θ with respect to an inertial frame.

b) Influence of disturbances T_d

$$\theta = \frac{1+TS}{IS^2(1+TS)+A(K_pS+K)} \cdot T_d \tag{4.4}$$

The stability of the system is studied by the Routh criterion. The characteristic equation of θ is

$$ITS^3 + IS^2 + AK_pS + AK = 0$$

S^3	IT	AK_p
S^2	I	AK
S^1	AK_p-TAK	0
S^0	AK	

The system will be stable if $K_p - TK > 0$, i.e. $\frac{K_p}{K} > T$. (In practical case $\frac{K_p}{K} = \lambda$ is of 10 to 20). The study of (4-4) is difficult enough due to the 3rd degree denominator. If yet it's assumed that the delay $\frac{1}{1+TS}$ is small enough i.e., $T = 0$ we get a greatly simplified expression:

$$\theta = \frac{1}{AK\left(\frac{I}{AK}S^2 + \frac{K_p}{K} S + 1\right)} T_d$$

The study is then straight-forward, and can be found in any feedback litterature (see for-instance Ref. 3). Such a study can only give an idea of the values for $K_1 K_p$, T etc.; for the approximations to get (4.4) for instance are not always justified. In particular, in (4.4) there is no mention of the coupling of the flywheels; and yet it's evident that such wheels behave as gyroscopes. H. Cannon in the ARS Journal (See Ref. 3) has studied an interesting way of coupling the three flywheels so that their momenta can be exchanged as the satellite rotated around the earth and so a great amount of energy can be spared.

As the Routh criterium has shown that the system was stable (if $K_p > T.K$), the flywheel stabilization will be useful for small and high frequency perturbations (where the gas rocket may be destabilized).

Both systems will be needed for a good stabilization; the flywheel system furnishings the damping of the perturbations already damped by the gas rocket system.

CHAPTER V

Conclusion

In conclusion it can be said that in order to achieve a good stabilization of the satellite, two corrective systems are needed. One, in most cases a gas rocket stabilization system, to correct strong perturbations torques ("strong", meaning here: of the order of some hundreds dyne x cm).

Such a system is also very interesting for impulse disturbances; and the phase plane method is very adequate to study the system under such perturbations. In fact, in the study done here, all the disturbances have been reduced to impulse disturbances by assuming that step disturbances and sinusoidally varying disturbances are small and of large periods; then they build up θ and $\dot{\theta}$ very slowly (due also to the large inertial of the satellite); and the gas rocket system will act only when θ and $\dot{\theta}$ (when the satellite was only under the influence of disturbances torques.) Such a method - reducing all disturbance torques to an impulse is only valid for slowly varying and small perturbations. It's justified for most of the perturbations encountered in space. But an interesting study could be done on the influence of a high frequency perturbation on the gas rocket system, or, in other words, under what conditions may the satellite be destabilized; then, the method of the first harmonic should probably be used instead of the phase plane method. Also for low-level disturbances (for instance some dyne x cm) the coupling between the 3 axes cannot be neglected; and the study of the gas rocket system, with coupling between the 3 axes of stabilization, is quite impossible without any mechanical means of computation. Also low-level disturbances need not be studied due to the fact that a gas rocket system is ineficient for the correction of a low level disturbance because of its limit cycle, therefore, a flywheel stabilization system has been studied for small disturbances.

In that case the system is a linear one; but the coupling between the 3 axes of stabilization is difficult to neglect and must be thoroughly examined. The flywheel system should be studied by means of a computer. Otherwise the equations, though not impossible to solve, become rather difficult. An interesting feature of a flywheel system is that a coupling between the 3 flywheels allow a great economy of power; for the energy can be transferred from one flywheel to another without too much losses. Such coupling has been studied by H. Cannon in the ARS Journal (see Ref. 3). For, if the complete study of the stabilization of a satellite is difficult, many separated studies can be entirely worked and without too much trouble, even without the use of a computer.

APPENDIX

4. Derivation of the transfer function for a flywheel

A diagram of such wheel is:

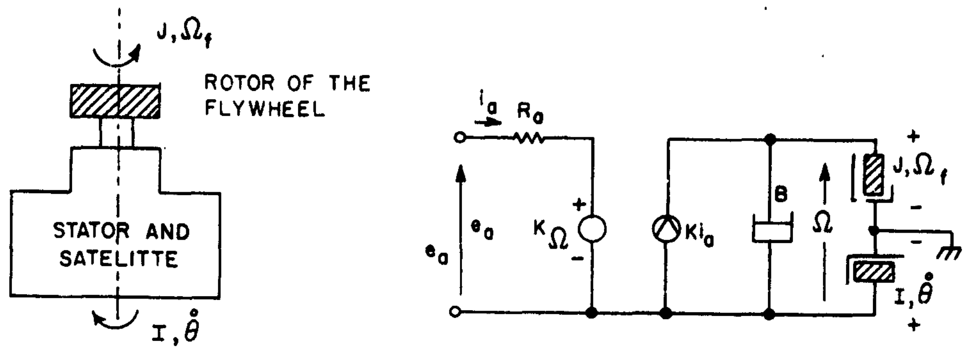


Fig. 4. I A Flywheel System

Assuming $\Omega_f - \dot{\theta} = \Omega$

$$\frac{IJ}{I+J} \sim J$$

it turns out the following relation:

$$\frac{e_a - K\Omega}{R_a} = i_a \qquad (1)$$

$$\frac{\Omega}{\dfrac{\frac{1}{BJS}}{\frac{1}{B} + \frac{1}{JS}}} = Ki_a$$

or

$$\Omega\,(JS + B) = Ki_a \qquad (2)$$

Equations (2) and (1) then yield:

$$\Omega\,(JS+B) = \frac{K}{R_a}\,(e_a - K\Omega)$$

or

$$\Omega\,(JS+B+\frac{K^2}{K_a}) = \frac{Ke_a}{R_a}$$

$$\frac{\Omega}{e_a} = \frac{\frac{K}{R_a}}{JS + (B + \frac{K^2}{R_a})} = \frac{A}{1+T_S}$$

with $$A = \frac{K}{K^2 + B\,R_a} \qquad T = \frac{JR_a}{BR_a + K^2}$$

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